Yellow Level

- Bilbo Baggins found some precious stones in Smaug's lair: diamonds, rubies and emeralds. The number of stones other than emeralds is 2 times less than the number of stones other than diamonds, and the number of emeralds is 90 more than the number of rubies. How many diamonds did Bilbo find?

Answer: 45.

Solution. Let us denote the number of diamonds by x and the number of rubies by y, therefore the number of emeralds is $(y + 90)$. The number of stones other than diamonds is 2 times more than the number of stones other than emeralds, therefore $y + (y + 90) = 2(x + y)$, from which $x = 45$.

1. What is the maximum number of sides of a figure that is a common part of a triangle and a convex quadrangle? Give an example, please.

Answer: 7.

Solution. Since these two figures are convex, their common part is also convex. Any two of its sides cannot lie on the same straight line. This means that any two sides of the common part are parts of different sides of these two figures, i.e., the common part has no more than $3 + 4 = 7$ sides.

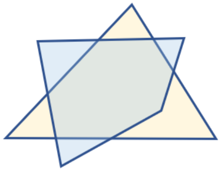

Fig. 1

The example is shown on Fig. 1 (see on the right).

Instruction for checking solutions: any reasoning similar to the mentioned solution without the first sentence (without referring to convexity) can be considered a complete solution.

2. Winnie-the-Pooh's barrel contained 22 kg of honey and N kg of peanut butter. After adding 15 kg of peanut butter to the barrel, its content in the barrel increased by 33%. What is N?

Answer: 3.

Solution. Let the barrel originally contained X% of peanut butter. Two correlations follow from the statement: $N = (22 + N) \cdot X\% : 100\%$ and $N + 15 = (22 + 15 + N) \cdot (X + 33)\% : 100\%$. This means that $X = 12$, $N = 3$.

Instruction for checking solutions. If the solution contains the answer only and explains that it satisfies the requirements of the statement, then such solution can be considered as partially correct only.

3. We found two divisors of 160 000. Their sum was equal to 1025. What was the larger of these divisors? Find all possible solutions.

Answer: 625 or 1000

Note: the pairs 1000 and 25, 625 and 400 satisfy the requirements of the statement.

Solution. Note that $160\,000 = 2^8 \cdot 5^4$. The sum 1025 is odd, and this means that one of the divisors is a power of 5. It is enough to try all options for which one of the divisors is 5, 25, 125, 625 to make sure that only two these options satisfy the requirements of the statement.

Instruction for checking solutions: if only one of the two options is mentioned, such solution cannot be considered complete.

4. Find the numerical value of $\left(\sqrt{12}+5\sqrt{3}\right)\left(\sqrt{578}-3\sqrt{8}\right)\cdot\sqrt{6}$.

5. Answer: 462.

Solution.
$\left(\sqrt{12}+5\sqrt{3}\right)\left(\sqrt{578}-3\sqrt{8}\right)\cdot\sqrt{6}=\left(2\sqrt{3}+5\sqrt{3}\right)\left(17\sqrt{2}-6\sqrt{2}\right)\cdot\sqrt{6}=7\sqrt{3}\cdot 11\sqrt{2}\cdot \sqrt{6}=77\cdot 6=462$

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Instruction for checking solutions. If the correct answer is not accompanied with calculations, such solution cannot be considered correct.

6. Let a natural number be called a 'number of interest', if for any natural k meeting the conditions $1 < k < 8$ either this number is divisible by k, or its digits can be rearranged so that the resulting number will be divisible by k. Find the smallest number of interest.

Answer 102.

Solution. Neither two-digit number satisfies the requirements of the statement. Indeed, if 0 is one of those digits, the number is be a multiple of the LCM $(2, 3, 4, 5, 6, 7) = 4\cdot 3\cdot 5\cdot 7 > 100$, but this is impossible. Otherwise, a number divisible by 5 should consist of an even digit and 5. But the numbers 25, 45, 65, 85 don't meet the requirements: a multiple of 3 cannot be obtained from 25, 65 and 85, and a multiple of 7 cannot be obtained from 45.

The numbers 100 and 101 don't meet the requirements: a multiple of 3 cannot be obtained from them.

The number 102 meets the requirements: 120 is a multiple of 2, 3, 4, 5, 6, and the number 210 is a multiple of 7.

Instruction for checking solutions. The solution has to contain the explanation for minimality of the number 102 and the explanation for the fact that 102 meets the requirements. If only one explanation is available, then such solution can be considered as partially correct only.

7. In a right triangle, the sum of its sides is 70 and the sum of the squares of its sides is 1682. Find the square of the difference of its legs.

Answer: 1.

Solution. It follows from the statement that $a+b+c=70$, $a^2+b^2+c^2=1682$, $a^2+b^2=c^2$. This means that $c^2=841$, $c=29$. Therefore, $a+b=41$, $a^2+b^2=841$, from which we get $(a-b)^2=2(a^2+b^2)-(a+b)^2=1$.

Instruction for checking solutions. An example of the triangle with sides of specific lengths meeting the requirements of the statement cannot be considered correct if it does not contain generalities.

8. Each cell of a 7×7 table contains a natural number. In any 1×3 or 3×1 part of this table, the sum of the numbers contained in its cells is 7. Can the sum of all numbers contained in the table be equal to 120?

Answer: no.

Solution. Any cell of the table cannot contain a number exceeding 5, otherwise in any 1×3 part containing this cell the sum of the numbers will exceed $5 + 1 + 1 = 7$. A 7×7 table can be split into sixteen 1×3 parts and one cell. Therefore, the sum of the numbers in the table does not exceed $16 \cdot 7 + 5 = 117$.

Note. See Fig. 2 on the right; it shows the arrangement of the numbers in the cells for which the requirement of the statement is met and the sum of which is 117.

5	1	1	5	1	1	5
1	1	5	1	1	5	1
1	5	1	1	5	1	1
5	1	1	5	1	1	5
1	1	5	1	1	5	1
1	5	1	1	5	1	1
5	1	1	5	1	1	5

Fig. 2

Instruction for checking solutions. If a solution contains the example with the sum 117, but does not contain the proof of maximality of this sum, then such solution can be considered as partially correct only.

9. There are several rooks on the chessboard. A study composer wants to paint each rook in any of N colours to avoid the situation when two rooks of the same colour can take each other. What is the smallest N meeting such conditions for any arrangement of the rooks? Any rooks cannot take each other if a rook of other colour is placed between them.

Answer: 3.

Solution. One can see that it is impossible to repaint 5 rooks arranged as shown on Fig. 3 on the right in 2 colours subject to meeting the requirements of the problem.

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Л		Л

Fig. 3

For any arrangement of the rooks, they may be painted in 3 colours subject to painting them row by row from top to bottom, and from left to right in each row: in this case each rook to be repainted can be taken by not more than two repainted rooks.

Instruction for checking solutions. The solution has to contain an explanation for insufficiency of 2 colours and an algorithm for 3 colours. If only the explanation or only the algorithm is contained in such solution, it can be considered as partially correct only.

10. 100 men sat at the Round Table, Each of them is either a knight who always tells the truth, or a knave who always lies. 28 of them gave the answer 'Yes' to the question 'Is the neighbour to your left a knight?'. What was the maximum number of liars sitting at the Round Table?

Answer: 64.

Solution. The answer 'Yes' can be obtained from a knight who answers about a knight or from a knave who answers about a knave. If two knights or two knaves are neighbours at the Round Table, one of them leaves the Table. In each such case, the answer to the question do not change, but the number of the answers 'Yes' decreases by one. When 28 of 100 men leave the Table, only answers 'No' can be obtained; this means that half of those who remain sitting at the Table (their total number is $100 - 28 = 72$) are knights and half of them are knaves. Therefore, the number of knaves does not exceed $72 : 2 + 28 = 36 + 28 = 64$.

This option is possible. Let 36 knights and 36 knaves sit at the Table so that each knight sits between two knaves and each knave sits between two knights. Then let 28 knaves sit between two neighbours. After that, the answers 'Yes' will be obtained from 28 knaves only.

Instruction for checking solutions: if the correct example of arrangement of 64 knaves and 36 knights is only available, it can be considered as a partially correct solution only.

11. Kai divides a 10×10 square ice plate into 3 parts and calculates the perimeter of each part. All perimeters are equal to N. What is the maximum possible N?

Answer: 68.

Solution. The part having smaller area consists of no more than $[100 : 3] = 33$ squares. Let us build this part by attaching the squares one by one. After every attachment, the perimeter increases by no more than 2. Therefore, the perimeter of this part does not exceed $4 + 32 \cdot 2 = 68$.

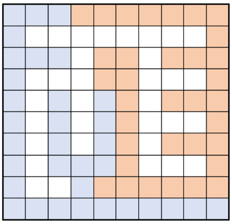

Fig. 4

One of the examples if shown on Fig. 4 on the right; in this case, the perimeter of each part is 68.

Instruction for checking solutions: if only the example is available (without the proof of maximality) or only the proof of maximality is available (without the example), the solution can be considered as partially correct only.